Journal of Sound and Vibration (2002) **254**(4), 777–786
doi:10.1006/jsvi.2001.4122, available online at http://www.idealibrary.com on IDEAL®

VIBRATIONAL MODES OF TRUMPET BELLS

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(*Received* 6 *August* 2001, *and in final form* 2 *November* 2001)

We report on an investigation of the normal modes of vibration of the bells of several modern trumpets. We describe the results of experiments using electronic speckle-pattern interferometry to visualize the modal structure and we show that the mode frequencies follow a generalized version of Chladni's law.

1. INTRODUCTION

Research into the physics of musical instruments has received much attention in the past few decades. Brass wind instruments have been of particular interest to many, and much progress has been made toward understanding the physics of trumpets, trombones, French horns, etc [1–4]. Naturally, most of the research concerning brass musical instruments has been directed toward understanding the dynamics of the air column, since most aspects of the acoustic signature of these instruments can be explained in terms of the macroscopic parameters associated with the air column. Significantly less attention has been directed toward understanding the effects of the perturbations on the ideal air column caused by vibrations of the walls and bell of the instruments.

The acoustic effects attributable to wall and bell vibrations of brass wind instruments are significantly less important than those effects attributable to the macroscopic features of the air column such as the shape of the instrument's bell. However, musicians and instrument makers almost universally claim that small changes in the material properties of the bell result in noticeable changes in the sound of the instrument. Within the scientific community, there is conflicting evidence concerning the importance of bell vibrations and the material parameters that may affect them [5–14].

If we are to understand the significance of the vibrations of the bell in the acoustic signature of brass instruments we believe that we must first understand the bell vibrations themselves, beginning with an understanding of the normal modes of vibration of the bell and the frequencies associated with them. In addition to assisting in understanding the acoustic significance of bell vibrations, an understanding of the vibrational characteristics may lend some insight into the bells themselves, possibly leading to some useful method of classification.

There have been limited experimental investigations into the nature of the vibrations of brass instrument bells in the past [7, 9, 11, 13, 14]. Most investigations published to date appear to have been conducted on the bells of trombones; however, it is likely that to within a reasonably good approximation the bells of all brass musical instruments have similar characteristics. This being the case, we have investigated the vibrational characteristics of trumpet bells and we believe that our fundamental conclusions may be extrapolated to the bells of other brass wind instruments.

0022-460X/02/$35.00

Here, we report on an experimental investigation of the vibrational modes of the bells of several modern trumpets. Among other things, we report that that the modes are well described by Bessel functions projected onto the bell; that knowing the frequencies of only a few of the modes allows one to predict the remaining modes without recourse to extensive mathematical analysis; and that trumpet bells can be characterized by three parameters unique to each bell.

2. EXPERIMENT

In order to investigate the vibrational modes of modern trumpets, we observe the bell vibrations in real time using electronic speckle-pattern interferometry (ESPI) [15]. The trumpet under investigation is mounted on a vibration-isolated optical table inside an anechoic chamber. The laser used to illuminate the trumpet is a continuous-wave, frequency-doubled, solid-state laser with a wavelength of 532 nm. The laser is mounted on a vibration-isolated table outside of the chamber and the light enters the chamber through a small hole in the wall.

The ESPI system is constructed from discrete optical components that are mounted on the same optical table as the musical instrument. The beam enters the chamber and is then split into two beams using a half-wave plate and a polarizing beam splitter. One beam, termed the object beam, is directed toward the instrument under study. The second beam, termed the reference beam, is directed toward a delay leg. Light from the object beam is reflected off the instrument under study and collected by a lens system. The lens system is designed to image the instrument under study onto a two-dimensional CCD array; however, a beam splitter is inserted between the final lens of the system and the CCD array. After traversing the delay leg the reference beam is spatially filtered, enlarged, and projected onto the CCD array via the beam splitter within the lens system. The image of the interference pattern created by the two beams on the CCD array is transferred to a computer outside of the chamber where real-time image subtraction produces an image of the instrument, where only the vibrating portions are visible. Since the image capture system operates at a standard video rate of 30 Hz and the lowest vibrational mode of most modern trumpets appears to be well above this frequency, the ESPI image is always the result of an average of many oscillations of the bell.

In contrast to some previously reported investigations of the bells of brass instruments, our investigations of bell vibrations are performed with the bell attached to the instrument. In our experiments, we excite the bell vibrations by attaching a small speaker to the mouthpiece of the instrument and drive the oscillations with a high-quality signal generator. Several experiments have shown that the bell vibrations are independent of the resonances of the air column of the instrument, and indeed similar results are obtained regardless of the configuration of the instrument's valves.

Typical results from our ESPI experiments are shown in the right three columns in Figure 1 (the first column will be addressed later). In the second column of Figure 1, the bell of a modern trumpet is shown end-on as it is vibrating in several of its normal modes. The light areas occur where the bell is vibrating; the dark areas indicate nodes. The third and fourth columns are images of the same bell driven at the same frequency viewed from the back and side respectively. By viewing the bell from three different perspectives, we can unambiguously determine the mode structure and classify the mode in the usual manner by designating the number of nodal meridians and nodal circles. Using this notation, the bell in first row of Figure 1 is vibrating in the (2,1) mode. Using ESPI in this manner, we have visually determined the normal-mode vibrational frequencies and patterns of several modern trumpets.

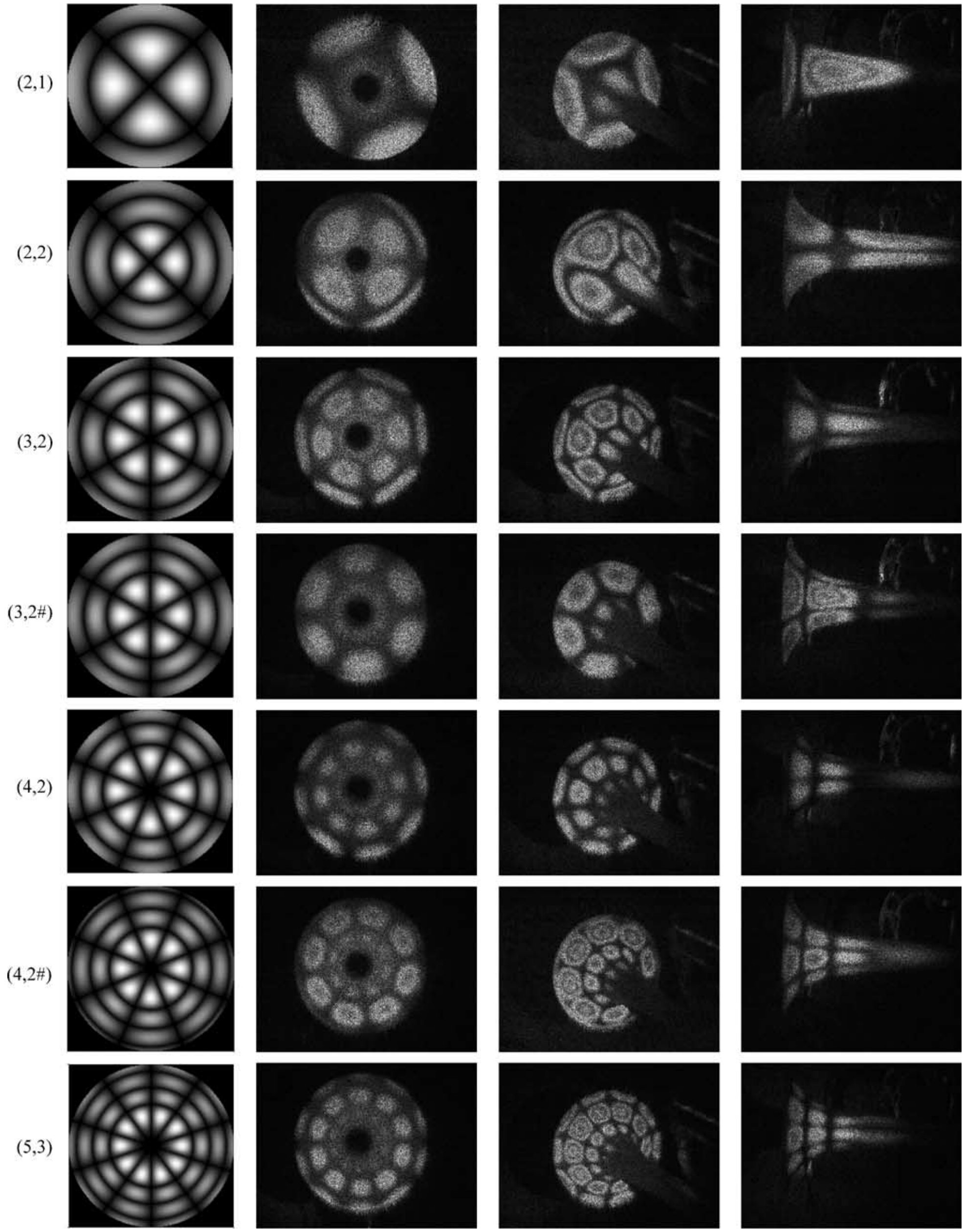

Figure 1. ESPI images of several normal modes of a modern trumpet bell alongside the calculated vibrational pattern for each. The first frame is the calculated mode pattern, the following frames are ESPI images of the front, back and side of the bell vibrating in that mode. In each case, the value of σ necessary to calculate the correct pattern was determined by visually matching the computed modal structure with the ESPI images. The value of σ in each case was determined to be $0{\cdot}32 \pm 0{\cdot}01$.

Before progressing to an analysis of the vibrational patterns of the trumpet bell, it is useful to note some general observations that we have made after studying the bells of several different trumpets. These observations derive from analyzing the bells of trumpets ranging in quality from extremely poor to professional.

The first general observation is that most of the vibrational modes that are easily excited appear to be circularly symmetric. Also, most of the modes that can be excited in all bells are identical, however, there are some exceptions and not all modes can be excited in all trumpet bells. Observations of many modal patterns on many different bells indicate that there is no obvious boundary to the vibrations, but they seldom extend past the bell brace. Some modes extend only half-way between the rim of the bell and the bell brace, while some patterns extend all the way from the rim to slightly beyond the bell brace. Finally, we have found that most vibrational patterns rotate with a slight change in frequency as is common in structures with slightly imperfect circular symmetry.

3. THEORY

Since the bells of brass instruments are not described by a simple mathematical function, calculating the vibrational modes of actual bells has proven to be quite difficult. The equations of motion of the surface of the bells are not mathematically tractable, and there appear to be no closed-form solutions. This being the case, researchers have turned to finite element analysis in an attempt to understand the vibrations of actual bells with some success [7, 16]. We have chosen to approach the modelling of the bell vibrations using a perturbative approach instead.

From numerous observations of the modal patterns of the bells of many trumpets, we believe that the fundamental modal patterns are well described by Bessel functions projected onto the bell shape. Since the normal vibrational modes of flat circular plates are described by Bessel functions, we have searched for a description of the fundamental modes by modelling the trumpet bell as a distorted circular plate. Using this logic, we assume the normal-mode solutions are simply perturbed solutions of those of a flat circular plate.

The equations describing the vibrations of a flat circular plate are well known, as are their solutions [17]. Assuming that the plate is thin, the solutions to the equation of motion for flexural waves in a plate are combinations of ordinary and hyperbolic Bessel functions. The angular solutions are either sine or cosine functions. In the laboratory, we have been unable to isolate any mode that compares well with the solutions containing the hyperbolic Bessel functions. Indeed, all experimentally realizable normal modes compare very well with the Bessel functions of the first kind, and while we cannot eliminate the modified Bessel functions on theoretical grounds, based on experimental evidence we will not consider them in this analysis. We, therefore, take the ordinary Bessel functions as the pertinent part of the solution and write the amplitude of vibration as

$$A(r, \phi) = B\mathrm{J}_m(kr)\cos(m\phi), \tag{1}$$

where J_m is the Bessel function of order m, r is the radial co-ordinate, and k and B are constants.

Naturally, the choice of the cosine function over the sine function for the angular term is arbitrary. We assume that the quasi-degenerate modes referred to earlier (i.e., similar modes of different angular orientation separated in frequency by only a few Hertz) are indicative of the presence of both sets of modes.

The boundary conditions that are applicable to the bell of a brass wind instrument are not obvious, and may be unknowable in detail. Without knowledge of the appropriate boundary conditions, we cannot accurately calculate the allowable values of the constant k, and therefore predicting the fundamental frequencies of vibration appears to be an intractable problem. To overcome this problem, we have chosen to visually match the observed vibrational patterns to Bessel functions and determine the appropriate values of k.

Experimentally determining the value of k in this manner allows us to search for a functional form that may lead to a predictive ability and eventually to an understanding of the boundary conditions.

If the instrument bell were indeed a flat circular plate, the values of the constant k for all of the fundamental modes would obey the equation

$$k = \sigma\sqrt{f}, \qquad (2)$$

where σ is a constant, although only an analysis of the boundary conditions will reveal the correct values of the normal frequencies of vibration. We will address the prediction of the correct modal frequencies later; however, it is reasonable to assume that the dispersion relation for the flexural waves in the instrument bell will approximate equation (2). We, therefore, assume that the dispersion relation for the bell of a brass instrument may be a perturbation on equation (2) and can conveniently be written as

$$k = C(r, f)\sqrt{f},$$

where $C(r, f)$ is some function that is dependent upon both the frequency and the distance from the axis of the bell. This functional form for k enables the subtle distortion of the Bessel function with distance from the center of the bell as well as a perturbation on the normal dispersion relation.

4. ANALYSIS

As noted above, we have determined the frequencies and the modal patterns of several modern trumpet bells. By visually determining the number of nodal meridians, the order of the appropriate Bessel function is easily determined, and the value of k can be determined by visually matching the patterns to two-dimensional plots of Bessel functions.

Since the Bessel functions (with the modified wave number dependent upon both frequency and position) must be visualized as being projected onto the complex shape of the instrument bell there can be some ambiguity in determining the most accurate fit to the data. However, by comparing the three views of the bell with a two-dimensional plot of the modelling function it is possible to achieve an excellent fit to the data in every case.

An extensive modelling effort was undertaken to determine the form of the function $C(r, f)$ that resulted in the most accurate fit to the actual bell vibrations. Initially, a search was made to find the radial dependence of k. It was discovered that the best-fit to the data occurred when the function was independent of r. That is, an ordinary Bessel function of order m was determined to be an excellent fit to each of the fundamental mode patterns. Likewise, an excellent fit to the data was achieved when the wave number was dependent only upon the square root of the frequency. Thus, equation (2) is an accurate description of the relationship between the wave number k and the frequency; however, the value of σ is very different from that of a thin, flat, circular plate.

Figure 1 shows several of the modes of a trumpet along with the calculated vibrational pattern for each. The instrument in this case was described by an instrument retailer as a top-level amateur or low-level professional instrument. The value of σ was determined to be $0{\cdot}32 \pm 0{\cdot}01$. A search of the literature discloses that the relationship between the frequencies of these modes is not consistent with those of a flat plate that is either free, clamped, or simply supported at the edge.

Since it appears that trumpet bells can be modelled as flat, circular plates, it is logical to ask if the fundamental modes mimic those of flat plates in other ways. For example, it is well

TABLE 1

Typical mode structure and resonant frequencies of a modern trumpet bell. The modes are designated by the number of nodal meridians and nodal circles (m, n)

m	n	f (Hz)
2	1	537
2	2	1018
3	2#	1207
3	2	1416
4	2#	1854
4	2	2201
5	3#	2628
5	3	2986

known that the modal frequencies of flat plates follow Chladni's law, which states that the effect on the frequency of adding a node bisecting the axis of the bell (nodal diameter) is approximately equal to that of adding two circular nodes. Indeed, it has been shown that the modal frequencies of some curved percussion instruments follow Chladni's law quite well, and it has even been shown to apply to the vibrational modes of many bells [18–20]. In its most general form, Chladni's law can be written as

$$f_{mn} = \gamma(m + bn)^{\mu}, \tag{3}$$

where m and n are the number of nodal diameters and nodal circles, respectively, f_{mn} is the frequency of the mode (m, n), and γ, b and μ are constants. The value of μ and b are both approximately 2 for a flat plate.

The modes that can be excited within the playable frequency range of a typical B-flat trumpet are shown in Table 1. Note that most modes have two resonant frequencies associated with them, with the exceptions being the (2, 1) and (2, 2) modes. When two modes have the same number of nodal lines and circles but occur at significantly different frequencies, the position of the nodal circles with respect to the end of the bell differ between the two.

The presence of more than one frequency being associated with a single mode is familiar from the study of carillons, church bells and hand bells. In these instruments, only modes with a single nodal circle exhibit this redundancy and the modes with the node near the sound bow are denoted by the symbol #. For example, mode (3, 1) has three nodal meridians and one nodal circle near the waist of the bell while mode (3, 1#) has three nodal meridians and one nodal circle near the sound bow [20].

An examination of Figure 1 shows that the modes of a trumpet bell can also be divided into modes with antinodes near the rim of the bell and modes with nodes near the rim. Following the established notation used in bells, we have annotated modes with nodes near the rim of the bell with # and classify them as "shell-driven modes". Modes with antinodes near the rim of the bell will be referred to as "ring-driven modes". (The use of these terms may not be in keeping with their original meaning; however, in the interest of continuity with the rest of the community, we will use them to refer to the different modal structures.)

After establishing the mode numbers, equation (3) can be fit to the data from any given bell and a reasonable approximation to a linear relationship between the predicted and measured frequencies can be established. The data are fit to equation (3) by allowing γ, b and μ to be free parameters while minimizing the difference between the calculated frequencies and the measured frequencies. Typical results of fitting the modal frequencies of a trumpet

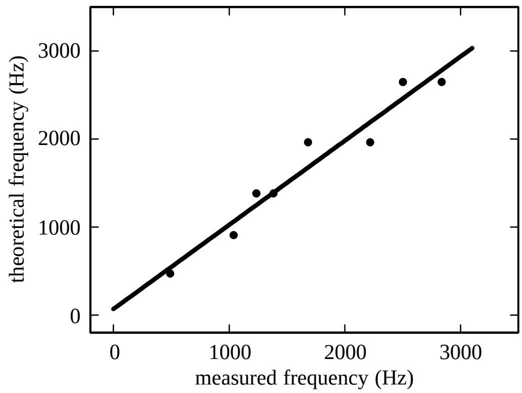

Figure 2. Typical results of fitting the modal frequencies of a trumpet bell to equation (3). The line results from a linear regression of the data. The symbols are larger than the uncertainty in the measurements.

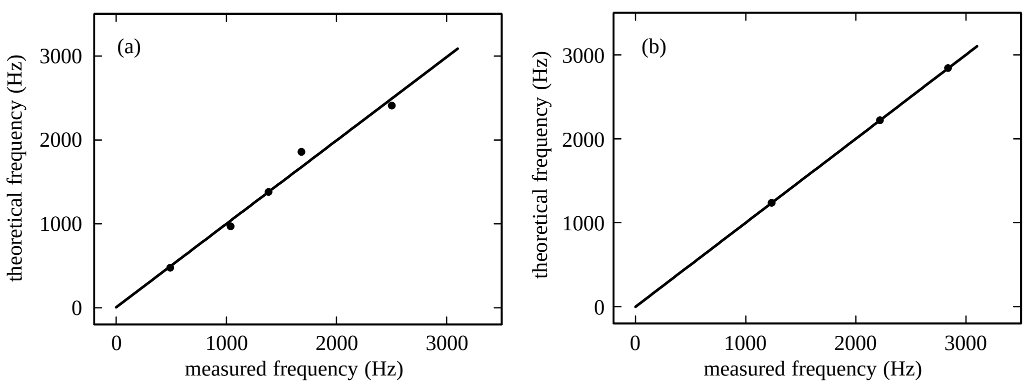

Figure 3. Typical results of fitting only the shell-driven modes (a) and only the ring-driven modes (b) to equation (3).The lines result from a linear regression of the data. The symbols are larger than the uncertainty in the measurements.

bell to equation (3) are shown in Figure 2, where the frequencies predicted by fitting the data to equation (3) are plotted versus the measured frequencies. The line resulting from performing a linear regression on the data is also shown in Figure 2. Note that while there is generally good agreement between the predictions of equation (3) and the experimental results, and the data indicate a significant measure of linearity, there is considerable deviation from linearity for several points. A close examination of the data indicates that the data can be collected into two independent groups that each follow Chladni's law quite well: one group is made up of modes with nodes near the rim (shell-driven modes) and one group is made up of modes with antinodes near the rim (ring-driven modes).

By separating the two types of modes prior to fitting the data to equation (3), an excellent agreement between Chladni's law and the data can be shown; Figure 3 demonstrates this. Figure 3(a) shows the frequencies of the shell-driven modes fit to equation (3) while Figure 3(b) shows the ring-driven modes fit to equation (3). The small number of ring-driven modes exhibited by this particular trumpet is typical and can lead one to question the importance of the linear fit to the data. However, we have observed one trumpet that exhibits a larger number of ring-driven modes than shell-driven modes. In this case, as in all

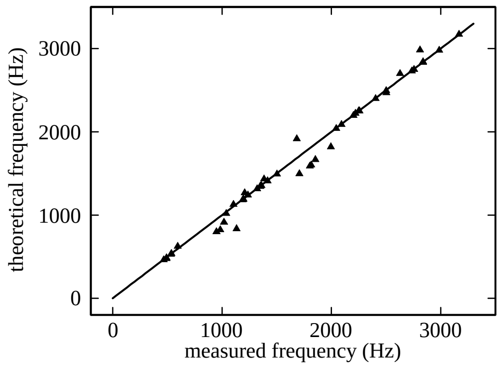

Figure 4. Data from six different trumpets fit individually to equation (3). The manufacturer and models are shown in Table 2 along with the fitting parameters. The line results from a linear regression of the data and has a slope of $1{\cdot}00 \pm 0{\cdot}02$. The symbols are larger than the uncertainty in the measurements.

TABLE 2

The parameters used to fit modal frequencies to Chladni's law for six different trumpets. Modes with nodes near the rim of the bell are termed shell-driven modes, while modes with antinodes near the rim are referred to as ring-driven modes

Model	γ (Hz)	b	μ
Shell-driven modes			
King legend	36	1·69	1·99
King silver flair	56	1·29	1·79
Conn 20B	71	2·12	1·51
Besson M800-2 (#1)	81	1·24	1·62
Besson M800-2 (#2)	97	0·95	1·58
Besson M800-2 (#3)	79	1·14	1·60
Ring-driven modes			
King legend	1927	− 1·31	0·45
King silver flair	800	− 0·678	1·07
Conn 20B	260	− 0·133	1·64
Besson M800-2 (#1)	324	− 0·106	1·44
Besson M800-2 (#2)	308	− 0·103	1·44
Besson M800-2 (#3)	312	− 0·114	1·42

cases, when a linear regression is performed on the frequencies of only the ring-driven modes, the slope of a line fit to the predicted versus measured frequencies is found to be unity with an uncertainty $<0{\cdot}2\%$.

Figure 4 shows the data from the bells of six different trumpets from varying manufactures. In each case the data from each instrument was individually fit to equation (3). The make and model of the trumpets are shown in Table 2 along with the values of the parameters γ, b and μ. A linear regression of the collective data reveals a slope of $0{\cdot}99 \pm 0{\cdot}01$.

When comparing the bells of different instruments, it is difficult to quantify the important physical differences in a meaningful way. This difficulty is primarily due to the fact that much of the manufacturing process is accomplished by hand. Also, there are significant variations in manufacturing technique between models and manufacturers. Finally, there is no known mathematical description for the curves of the mandrels used to form the bells. The shapes of these mandrels are individually designed and often vary only slightly between models. The characterization is complicated by the different techniques that are used to form the bells and attach them to the rest of the instrument.

The thickness of the metal used to form the bell is clearly an important parameter in determining the vibrational characteristics; however, in the shaping process the metal is deformed to such an extent that the thickness can vary significantly with position. One easily measured parameter that is useful for the purposes of identification is the diameter of the bell at the rim. This information is routinely provided by the manufacturer; however, we have found that many bells with significantly varying vibrational characteristics have identical rim diameters.

All of the bells listed in Table 2 have a nominal wall thickness of 0·020 in. The rim diameters of the bells of the instruments shown in Table 2 are 4·875 in except for the Besson model, which has a bell diameter of 4·75 in. Given the similar physical measurements and the widely varying parameters required to fit equation (3), we believe that the most important aspect of the bell in determining the modal structure is the shape of the bell, which as noted above is extremely difficult to quantify.

Given the variation in manufacturing details between instruments it is logical to ask if the values of γ, b and μ can in some way be used to identify and characterize individual trumpet bells. In order for such a characterization to be useful, the values of these parameters must be relatively constant among similar bells. That is, the parameters γ, b and μ must be sensitive enough to characterize individual bell shapes, but insensitive enough so that the values are similar for bells with only very minor variations. In order to test the sensitivity of these parameters, three trumpets of the same model were evaluated. An examination of Table 2 reveals that the parameters associated with the three trumpets are very similar to each other and dissimilar to all of the other trumpets tested. These results indicate that it may be possible to use the parameters γ, b and μ in some way to characterize and identify bells, and it is logical to attempt to define a single parameter that will uniquely characterize each bell. From our study, it appears that the value of either γ or μ may be significant enough to uniquely characterize a specific bell design.

The parameter γ is the theoretical frequency of the $(1, 0)$ mode according to equation (3), and it is conceivable that this parameter is so dependent upon the fundamental design of the bell that it will suffice as a characterization of the bell. Unfortunately, to give any of these parameters meaning it is necessary to understand the effect that changing individual properties of the bell has on their value; we currently lack this understanding and until a much more detailed study can be performed, we will not attempt to capture the uniqueness of a bell in a single parameter.

5. CONCLUSIONS

In conclusion, we have shown that the fundamental vibrational modes of the bells of trumpets, and presumably many other brass wind instruments, are very similar to those of a flat circular plate. While the complex boundary conditions make it difficult to predict the resonant frequencies of the bells, these frequencies can be accurately described using a general form of Chladni's law.

We have also proposed that the parameters associated with fitting the modal frequencies to Chladni's law may be sufficient to uniquely characterize the individual bells of brass wind instruments. Specifically, the predicted value of the frequency of the (1,0) mode may characterize any particular bell design.

We have not addressed how the bell vibrations of brass wind instruments affect the acoustic signature of the instrument. We have merely attempted to characterize and understand the vibrational characteristics of these complex structures. Having determined the fundamental nature of the vibrational modes of brass instrument bells, we are currently attempting to understand how the bell vibrations enter into the larger picture of the musical acoustics of brass wind instruments.

ACKNOWLEDGMENTS

This work was supported by a grant from the Jessie Ball duPont Fund. The authors wish to thank G. Underwood and S. Tobin for the generous donation of instruments.

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